

HOMOTOPY DECOMPOSITION OF DIAGONAL ARRANGEMENTS

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ABSTRACT. Given a space X and a simplicial complex K with m -vertices, the arrangement of partially diagonal subspaces of X^m , called the dragonal arrangement, is defined. We decompose the suspension of the diagonal arrangement when $2(\dim K + 1) < m$, which generalizes the result of Labassi [L]. As a corollary, we calculate the Euler characteristic of the complement $X^m - \Delta_K(X)$ when X is a closed connected manifold.

1. INTRODUCTION AND STATEMENT OF RESULTS

A homotopy decomposition is a powerful tool in studying the topology of subspace arrangements and their complements. Ziegler and Živaljević [ZZ] give a homotopy decomposition of the one point compactification of affine subspace arrangements, from which one can deduce the well known Goresky-MacPherson formula [GM]. Bahri, Bendersky, Cohen, and Gitler [BBCG] give a homotopy decomposition of polyhedral products, a generalization of coordinate subspace arrangements and their complements, after a suspension, from which one can deduce Hochster's formula on related Stanley-Reisner rings. A homotopy decomposition of polyedral products due to Grbić and Theriault [GT] and the authors [IK1, IK2] also implies the Golod property of several related simplicial complexes. In this paper, we consider a homotopy decomposition of diagonal arrangements which is defined as follows. Given a space X , we assign a partially diagonal subspace of X^m corresponding to a subset $\sigma \subset [m] = \{1, \dots, m\}$ as

$$\Delta_\sigma(X) = \{(x_1, \dots, x_m) \in X^m \mid x_{i_1} = \dots = x_{i_k} \text{ for } \{i_1, \dots, i_k\} = [m] - \sigma\}.$$

Throughout the paper, let K be a simplicial complex on the index set $[m]$, possibly with ghost vertices, where we always assume that the empty subset of $[m]$ is a simplex of K . We define the arrangement of partially diagonal subspaces of X^m as

$$\Delta_K(X) = \bigcup_{\sigma \in K} \Delta_\sigma(X),$$

which is called the diagonal arrangement associated with K . Since $\Delta_K(X)$ is actually the union of the partially diagonal subspaces $\Delta_F(X)$ for facets F of K , it is also called the hypergraph arrangement associated with the hypergraph whose edges are facets of K . Diagonal arrangements include many important subspace arrangements. For example, if K is the $(m-3)$ -skeleton of $(m-1)$ -simplex, $\Delta_K(X)$ is the braid arrangement of X . Topology and combinatorics of diagonal arrangements have been studied in several directions. See [Ko, PRW, Ki, KS, L, MW, M] for

2010 *Mathematics Subject Classification.* 52C35, 55P10.

Key words and phrases. diagonal arrangement, polyhedral product, homotopy decomposition.

example. We are particularly interested in the homotopy type of $\Delta_K(X)$. Labassi [L] showed that the suspension $\Sigma\Delta_K(X)$ decomposes into a certain wedge of smash products of copies of X when K is the $(m-d-1)$ -skeleton of the $(m-1)$ -simplex and $2d > m$, in which case $\Delta_K(X)$ consists of all $(x_1, \dots, x_m) \in X^m$ such that at least d -tuple of x_i 's are identical. The proof for this decomposition in [L] heavily depends on the symmetry of the skeleta of simplices, and then it cannot apply to general K . The aim of this note is to generalize this result to arbitrary K with $2(\dim K + 1) < m$ by a new method, where the result is best possible in the sense that if $2(\dim K + 1) \geq m$, the decomposition does not hold as is seen in [L].

Theorem 1.1. *If X is a connected CW-complex and $2(\dim K + 1) < m$, then*

$$\Sigma\Delta_K(X) \simeq \Sigma\left(\bigvee_{\sigma \in K} \widehat{X}^{|\sigma|} \vee \widehat{X}^{|\sigma|+1}\right)$$

where \widehat{X}^k is the smash product of k -copies of X for $k > 0$ and \widehat{X}^0 is a point.

As a corollary, we calculate the Euler characteristic of the complement of the diagonal arrangement $\mathcal{M}_K(X) = X^m - \Delta_K(X)$.

Corollary 1.2. *Let X be a closed connected n -manifold. If $2(\dim K + 1) < m$, the Euler characteristic of $\mathcal{M}_K(X)$ is given by*

$$\chi(\mathcal{M}_K(X)) = \chi(X)^m - (-1)^{mn} \chi(X) \left(1 + \sum_{\emptyset \neq \sigma \in K} (\chi(X) - 1)^{|\sigma|}\right).$$

Remark 1.3. Corollary 1.2 does not hold without compactness of X . For example, if $X = \mathbb{R}$ (hence $n = 1$) and K consists only of the empty subset of $[m]$, $\mathcal{M}_K(X)$ is the off-diagonal subset of \mathbb{R}^m which has the homotopy type of S^{m-2} . Then $\chi(\mathcal{M}_K(X)) = 1 + (-1)^m$, which differs from Corollary 1.2.

ACKNOWLEDGEMENT. The authors are grateful to Sadok Kallel for introducing the paper [L] to them.

2. PROOFS

Before considering the proof of Theorem 1.1, we prepare two lemmas on homotopy fibrations.

Lemma 2.1 ([F, Proposition, pp.180]). *Let $\{F_i \rightarrow E_i \rightarrow B\}_{i \in I}$ be an I -diagram of homotopy fibrations over a fixed connected base B . Then*

$$\operatorname{hocolim}_I F_i \rightarrow \operatorname{hocolim}_I E_i \rightarrow B$$

is a homotopy fibration.

Lemma 2.2. *Consider a homotopy fibration $F \xrightarrow{j} E \xrightarrow{\pi} B$ of connected CW-complexes. If $\Sigma j : \Sigma F \rightarrow \Sigma E$ has a homotopy retraction, then*

$$\Sigma E \simeq \Sigma B \vee \Sigma F \vee \Sigma(B \wedge F).$$

Proof. Let $r : \Sigma E \rightarrow \Sigma F$ be a homotopy retraction of Σj , and let ρ be the composite

$$\Sigma E \rightarrow \Sigma E \vee \Sigma E \vee \Sigma E \xrightarrow{\Sigma \pi \vee r \vee \Delta} \Sigma B \vee \Sigma F \vee \Sigma(E \wedge E) \xrightarrow{1 \vee 1 \vee (\pi \wedge r)} \Sigma \check{B}$$

where $\check{A} = A \vee F \vee (A \wedge F)$ for a space A . Since ΣE and $\Sigma B \vee \Sigma F \vee \Sigma(B \wedge F)$ are simply connected CW-complexes, it is sufficient to show that ρ is an isomorphism in homology by the J.H.C. Whitehead theorem. We first observe the special case that there is a fiberwise homotopy equivalence $\theta : B \times F \rightarrow E$ over B . Then it is straightforward to see

$$\rho_* \circ \theta_*(b \times f) = b \times \hat{\theta}_*(f) + \sum_{|b_i| < |b|} b_i \times f_i$$

for singular chains b, b_i in B and f, f_i in F , where we omit writing the suspension isomorphism of homology and $\hat{\theta}$ is a self-homotopy equivalence of F given by the composite

$$\Sigma F \xrightarrow{j} \Sigma(B \times F) \xrightarrow{\theta} \Sigma E \xrightarrow{r} \Sigma F.$$

This readily implies that the map $\rho \circ \theta$ is an isomorphism in homology, and then so is ρ . For non-connected B , the above is also true if we assume that r is a homotopy retraction of the suspension of the fiber inclusion on each component of B . We next consider the general case. Let B_n be the n -skeleton of B , and let $E_n = \pi^{-1}(B_n)$. We prove that the restriction $\rho|_{\Sigma E_n} : \Sigma E_n \rightarrow \Sigma \check{B}_n$ is an isomorphism in homology by induction on n . Since B is connected, j is homotopic to the composite

$$F \simeq \pi^{-1}(b) \xrightarrow{\text{incl}} E$$

for any $b \in B$. Then $\rho|_{\Sigma E_0} : \Sigma E_0 \rightarrow \Sigma \check{B}_0$ is an isomorphism in homology. Consider the following commutative diagram of homology exact sequences.

$$(2.1) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & H_*(E_{n-1}) & \longrightarrow & H_*(E_n) & \longrightarrow & H_*(E_n, E_{n-1}) \longrightarrow \cdots \\ & & \downarrow (\rho|_{\Sigma E_{n-1}})_* & & \downarrow (\rho|_{\Sigma E_n})_* & & \downarrow (\rho|_{\Sigma E_n})_* \\ \cdots & \longrightarrow & H_*(\check{B}_{n-1}) & \longrightarrow & H_*(\check{B}_n) & \longrightarrow & H_*(\check{B}_n, \check{B}_{n-1}) \longrightarrow \cdots \end{array}$$

By the induction hypothesis, $(\rho|_{\Sigma E_{n-1}})_*$ is an isomorphism. Since B_{n-1} is a subcomplex of B_n , there is a neighborhood $U \subset B_n$ of B_{n-1} which deforms onto B_{n-1} . By the excision isomorphism, there is a commutative diagram of natural isomorphisms

$$\begin{array}{ccccc} H_*(E_n, E_{n-1}) & \xrightarrow{\cong} & H_*(E_n, \pi^{-1}(U)) & \xleftarrow{\cong} & H_*(E_n - E_{n-1}, \pi^{-1}(U) - E_{n-1}) \\ \downarrow (\rho|_{\Sigma E_n})_* & & \downarrow (\rho|_{\Sigma E_n})_* & & \downarrow (\rho|_{\Sigma E_n})_* \\ H_*(\check{B}_n, \check{B}_{n-1}) & \xrightarrow{\cong} & H_*(\check{B}_n, \check{U}) & \xleftarrow{\cong} & H_*(\check{B}_n - \check{B}_{n-1}, \check{U} - \check{B}_{n-1}) \end{array}$$

where we may chose the basepoints of B_n and U in $U - B_{n-1}$ since B is connected. Since each connected component of $B_n - B_{n-1}$ is contractible, $E_n - E_{n-1}$ is fiberwise homotopy equivalent to $(B_n - B_{n-1}) \times F$ over $B_n - B_{n-1}$, and then so is also $\pi^{-1}(U) - E_{n-1}$ to $(U - B_{n-1}) \times F$ over $U - B_{n-1}$. As in the 0-skeleton case, we see that Σr restricts to a homotopy retraction of

the suspension of the fiber inclusion on each component of $\Sigma(B_n - B_{n-1})$. Then by the above trivial fibration case, we obtain that the map

$$(\rho|_{\Sigma(E_n - E_{n-1})})_* : H_*(E_n - E_{n-1}, \pi^{-1}(U) - E_{n-1}) \rightarrow H_*(\check{B}_n - \check{B}_{n-1}, \check{U} - \check{B}_{n-1})$$

is an isomorphism, hence so is the right $(\rho|_{\Sigma E_n})_*$ in (2.1). Thus by the five lemma, the middle $(\rho|_{\Sigma E_n})_*$ in (2.1) is an isomorphism. We finally take the colimit to get that the map ρ is an isomorphism in homology as desired, completing the proof. \square

Remark 2.3. If we assume further that F is of finite type, it immediately follows from the Leray-Hirsch theorem that the map ρ is an isomorphism in cohomology with any field coefficient, implying that ρ is an isomorphism in the integral homology by [H, Corollary 3A.7].

We now consider the diagonal arrangement $\Delta_K(X)$. Suppose that $2(\dim K + 1) < m$, or equivalently, $2|\sigma| < m$ for any $\sigma \in K$. Then for $(x_1, \dots, x_m) \in \Delta_K(X)$, there is unique $x \in X$ such that $x_{i_1} = \dots = x_{i_k} = x$ with $i_1 < \dots < i_k$ and $2k > m$. Then by assigning such x to $(x_1, \dots, x_m) \in \Delta_K(X)$, we get a continuous map

$$\pi : \Delta_K(X) \rightarrow X.$$

For $\tau \subset [m]$, let $X^\tau = \{(x_1, \dots, x_m) \in X^m \mid x_i = * \text{ for } i \in [m] - \tau\}$, and we put

$$X^K = \bigcup_{\sigma \in K} X^\sigma$$

which is called the polyhedral product or the generalized moment-angle complex associated with the pair $(X, *)$ and K . Observe that for $2(\dim K + 1) < m$, we have $\pi^{-1}(*) = X^K$.

Proposition 2.4. *If X is a CW-complex and $2(\dim K + 1) < m$, then $X^K \rightarrow \Delta_K(X) \xrightarrow{\pi} X$ is a homotopy fibration.*

Proof. For each $\sigma \in K$, the map $\pi|_\sigma : \Delta_\sigma(X) \rightarrow X$ is identified with the projection from the product of copies of X . Then it follows from Lemma 2.1 that

$$\operatorname{hocolim}_K X^\sigma \rightarrow \operatorname{hocolim}_K \Delta_\sigma(X) \rightarrow X$$

is a homotopy fibration. Since the inclusions $X^\sigma \rightarrow X^\tau$ and $\Delta_\sigma(X) \rightarrow \Delta_\tau(X)$ for any $\sigma \subset \tau \subset [m]$ are cofibrations, we have

$$\operatorname{hocolim}_K X^\sigma \simeq \operatorname{colim}_K X^\sigma = X^K \quad \text{and} \quad \operatorname{hocolim}_K \Delta_\sigma(X) \simeq \operatorname{colim}_K \Delta_\sigma(X) = \Delta_K(X),$$

completing the proof. \square

Put $\widehat{X}^K = \bigvee_{\emptyset \neq \sigma \in K} \widehat{X}^{|\sigma|}$. In [BBCG], it is proved that there is a homotopy equivalence

$$(2.2) \quad \epsilon_X : \Sigma X^K \xrightarrow{\sim} \Sigma \widehat{X}^K$$

which is natural with respect to X , i.e. for a map $f : X \rightarrow Y$, the square diagram

$$\begin{array}{ccc} \Sigma X^K & \xrightarrow{\epsilon} & \Sigma \widehat{X}^K \\ \Sigma f^K \downarrow & & \downarrow \Sigma \hat{f}^K \\ \Sigma Y^K & \xrightarrow{\epsilon} & \Sigma \widehat{Y}^K \end{array}$$

is homotopy commutative, where the vertical arrows are induced from f .

Proposition 2.5. *If X is a CW-complex and $2(\dim K + 1) < m$, the inclusion $j : X^K \rightarrow \Delta_K(X)$ has a homotopy retraction after a suspension.*

Proof. Let $E : X \rightarrow \Omega \Sigma X$ be the suspension map. Since ΣE has a retraction, we easily see that the induced map $\Sigma \widehat{E}^K : \Sigma \widehat{X}^K \rightarrow \Sigma \widehat{\Omega \Sigma X}^K$ has a retraction, say r . If Y is an H-space, the map

$$Y \times Y^K \rightarrow \Delta_K(Y), \quad (y, (y_1, \dots, y_m)) \mapsto (y \cdot y_1, \dots, y \cdot y_m)$$

is a map between homotopy fibrations with common base and fiber, and then is a weak homotopy equivalence. Hence if Y has the homotopy type of a CW-complex, the map is a homotopy equivalence, implying that there is a homotopy retraction $r' : \Delta_K(Y) \rightarrow Y^K$ of the inclusion $j : Y^K \rightarrow \Delta_K(Y)$. Combining the above maps, we get a homotopy commutative diagram

$$\begin{array}{ccccccc} \Sigma \widehat{X}^K & \xlongequal{\quad} & \Sigma \widehat{X}^K & \xrightarrow{\epsilon^{-1}} & \Sigma X^K & \xrightarrow{\Sigma j} & \Sigma \Delta_K(X) \\ \parallel & & \downarrow \Sigma \widehat{E}^K & & \downarrow \Sigma E^K & & \downarrow \Sigma \Delta_K(E) \\ & & \Sigma(\Omega \Sigma X)^K & \xrightarrow{\Sigma j} & \Sigma \Delta_K(\Omega \Sigma X) & & \\ & & \parallel & & \parallel & & \\ \Sigma \widehat{X}^K & \xleftarrow{r} & \Sigma \widehat{\Omega \Sigma X}^K & \xleftarrow{\epsilon} & \Sigma(\Omega \Sigma X)^K & \xleftarrow{\Sigma r'} & \Sigma \Delta_K(\Omega \Sigma X) \end{array}$$

where $\Delta_K(E) : \Delta_K(X) \rightarrow \Delta_K(\Omega \Sigma X)$ is induced from E . Thus the composite

$$\Sigma \Delta_K(X) \xrightarrow{\Sigma \Delta_K(E)} \Sigma \Delta_K(\Omega \Sigma X) \xrightarrow{\Sigma r'} \Sigma(\Omega \Sigma X)^K \xrightarrow{\epsilon} \Sigma \widehat{\Omega \Sigma X}^K \xrightarrow{r} \Sigma \widehat{X}^K \xrightarrow{\epsilon^{-1}} \Sigma X^K$$

is the desired homotopy retraction. \square

Proof of Theorem 1.1. If $2(\dim K + 1) < m$, there is a homotopy fibration $X^K \rightarrow \Delta_K(X) \rightarrow X$, where the fiber inclusion has a homotopy retraction after a suspension by Proposition 2.5. Then by Lemma 2.2, we get a homotopy equivalence

$$\Sigma \Delta_K(X) \simeq \Sigma X \vee \Sigma X^K \vee \Sigma(X \wedge X^K).$$

Therefore the proof is completed by (2.2). \square

Proof of Corollary 1.2. Since X is a compact manifold, $\Delta_K(X)$ is a compact, locally contractible subset of an mn -manifold X^m . Then by the Poincaré-Alexander duality [H, Proposition 3.46], there is an isomorphism

$$H_i(X^m, \mathcal{M}_K(X); \mathbb{Z}/2) \cong H^{mn-i}(\Delta_K(X); \mathbb{Z}/2),$$

implying that $\chi(X^m, \mathcal{M}_K(X)) = (-1)^{mn} \chi(\Delta_K(X))$. Thus since $\chi(\widehat{X}^k) = (\chi(X) - 1)^k + 1$ for $k \geq 1$, it follows from Theorem 1.1 that

$$\chi(X^m, \mathcal{M}_K(X)) = (-1)^{mn} \chi(X) \left(1 + \sum_{\emptyset \neq \sigma \in K} (\chi(X) - 1)^{|\sigma|}\right).$$

Therefore the proof is completed by the equality $\chi(X^m) = \chi(X^m, \mathcal{M}_K(X)) + \chi(\mathcal{M}_K(X))$. \square

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